This document has some practice questions to help you prepare for the midterm.

1 Derivatives

1.1 Differentiation

1. Use the limit definition to compute the derivative of $f(x) = \frac{x+1}{2-x}$.

2. Use the limit definition to compute the derivative of $f(x) = \cos(3x)$.
   Try to simplify your answer as much as possible, you may need this trigonometric identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.

3. Use the limit definition to compute the derivative of $f(x) = \frac{a}{x}$.

4. Which of the following functions does not have a **continuous** derivative?
   a) $x^2 - 2x + 1$, b) $\cos(x^2)$, c) $|x - 1|$

1.2 Total derivative

1. Compute the total derivative of the Beattie-Bridgeman equation of state for the pressure,

$$p(T, \nu) = \frac{RT}{\nu^2}\left(1 - \frac{c}{\nu T^3}\right)(\nu + B) - \frac{A}{\nu^2}, \tag{1}$$

   where A and B are some functions of the molar volume ν, here we will consider them as constants.

2. What is the total derivative of a linear model of the form,

$$f(\mathbf{x}) = \sum_{i=1}^{m} w_i x_i, \quad \text{where} \quad \mathbf{x} = [x_1, x_2, \cdots, x_m]. \tag{2}$$

3. What is the total derivative of a polynomial expansion of the form,

$$f(x) = \sum_{i=1}^{m} w_i x^i \quad \text{where} \quad x^i \quad \text{is x to the power of } i. \tag{3}$$

1.3 Partial derivatives

1. The virial expansion is a model of thermodynamic equations of state, and it is given by,

$$P(V_m, T) = \frac{RT}{V_m}\left(1 + \frac{B(T)}{V_m} + \frac{B(T)}{V_m^2} + \cdots\right), \tag{4}$$

where V_m is the molar volume.
Compute the partial derivate of $P(V_m, T)$ with respect to the molar volume.

2. Compute the partial derivative of $f(x, y) = (2x + 6y)^4$ with respect to the variable y

$$\frac{\partial f(x, y)}{\partial y} = \frac{\partial}{\partial y}(2x + 6y)^4 \tag{5}$$

3. Compute the partial derivative of $f(x_1, x_2) = \frac{e^{x_1}}{e^{x_1} + e^{x_2}}$ with respect to the variable x_1

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{e^{x_1}}{e^{x_1} + e^{x_2}}\right) \tag{6}$$

2 Integrals

2.1 Substitution

1. Compute the indefinite integral,

$$\int x^3 \left(1 - 12x^4\right)^{1/8} \, dx. \tag{7}$$

2. Compute the indefinite integral,

$$\int \frac{x}{\sqrt{8 - 2x^2}} \, dx. \tag{8}$$

3. Compute the indefinite integral,

$$\int 7x^4 e^{x^5} \, dx. \tag{9}$$

4. Compute the indefinite integral,

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx. \tag{10}$$

Cont.

2.2 Rational functions and logarithms

1. Compute the indefinite integral,

$$\int \frac{x+5}{x}\,dx. \tag{11}$$

2. Compute the indefinite integral,

$$\int \frac{x}{x+5}\,dx. \tag{12}$$

Try to rewrite the integrand before integrating, you may need to write $\frac{x}{x+5} = 1 - \frac{5}{x+5}$.

3. Compute the indefinite integral,

$$\int \frac{1}{x\ln(x)}\,dx. \tag{13}$$

2.3 Trigonometric integrals

1. Compute the indefinite integral,

$$\int \cos^2(x)\sin(x)\,dx. \tag{14}$$

2. Compute the indefinite integral,

$$\int \frac{\cos(x)}{\sin^3(x)}\,dx. \tag{15}$$

3. Compute the indefinite integral,

$$\int \sec^9(x)\tan(x)\,dx. \tag{16}$$

Try to simplify the integrand first, you may need the derivative $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$.

3 Thermodynamics

3.1 Ideal gas

In class, we showed that the heat capacities are defined by

$$C_P = \left(\frac{dq}{dT}\right)_p, \qquad C_V = \left(\frac{\partial U}{\partial T}\right)_V.$$

Consider a closed system containing n moles of **ideal gas** undergoing an expansion at constant pressure, with only pressure–volume work, $dW = -pdV$.

Cont.

1. Using the first law, show that

$$C_P = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p.$$

Hint: Write the total differentials of $U = U(T, p)$ and $V = V(T, p)$, and apply the constant-pressure condition.

2. For an ideal gas, use your result from part 1 to show that

$$C_P = C_V + nR.$$

Hint: For an ideal gas the internal energy depends only on temperature.

3.2 Excluded-volume gas

Consider n moles of a gas described by

$$p(V - nb) = nRT,$$

where b is a positive constant representing the molar volume excluded by the molecules. Assume that the internal energy depends only on temperature, $U = U(T)$.

1. Using the relationship

$$C_P = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p,$$

show that this gas satisfies

$$C_P - C_V = nR.$$

Hint: Express V as a function of T and p, and recall that $C_V = (\partial U/\partial T)_V$.

2. Does the relationship $C_P - C_V = nR$ necessarily imply that a gas is ideal? Explain using your result.

3.3 Incompressible system

Consider an idealized liquid or solid whose volume is independent of temperature and pressure. Assume that its internal energy depends only on temperature, $U = U(T)$.

1. Using the relationship

$$C_P = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p,$$

show that

$$C_P = C_V.$$

Hint: Consider how the volume changes when the system is heated at constant pressure.

2. Explain physically why the heat capacities are equal, referring to the first law and pressure–volume work.

The End.