Practice questions

1. Compute each $A\,\mathbf{x}$ by dot products of the rows with the column vector:

   (a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

2. (a) What is the 2 by 2 identity matrix? $\mathbf{I}$ times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} x \\ y \end{bmatrix}$.

   (b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} y \\ x \end{bmatrix}$.

   (c) What 2 by 2 matrix R rotates every vector by 90°? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} y \\ -x \end{bmatrix}$.

   (d) What 2 by 2 matrix R^2 rotates every vector by 180°?

3. Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

4. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DC \qquad A(B+C).$$

5. Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

6. In the previous problem, multiply A times BC. Then multiply AB times C.

7. Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I. The only matrices that commute with B and C and all other 2 by 2 matrices are $A =$ multiple of I.

8. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$. Compute $\mathbf{u}^{\mathrm{T}}\,\mathbf{v}$, $\mathbf{v}^{\mathrm{T}}\,\mathbf{u}$, $\mathbf{u}^{\mathrm{T}}\,\mathbf{u}$ and $\|\,\mathbf{u}\,\|$.

9. Find all values of c for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ c \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

10. For $\mathbf{u} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}3\\4\end{bmatrix}$, compute the inner product $\mathbf{u}^{\mathrm{T}}\,\mathbf{v}$ and the outer product $\mathbf{u}\,\mathbf{v}^{\mathrm{T}}$. Give the shape of each result, and show that the trace of $\mathbf{u}\,\mathbf{v}^{\mathrm{T}}$ equals $\mathbf{u}^{\mathrm{T}}\,\mathbf{v}$.

11. Let $A = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, $\mathbf{u} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}$. Verify that

$$(A\,\mathbf{u})^{\mathrm{T}}\,\mathbf{v} = \mathbf{u}^{\mathrm{T}}(A^{\mathrm{T}}\,\mathbf{v}).$$

12. Suppose $\mathbf{u}^{\mathrm{T}}\,\mathbf{v} = 0$ and $\mathbf{u}^{\mathrm{T}}\,\mathbf{w} = 0$. Show that $\mathbf{u}^{\mathrm{T}}(\mathbf{v} + 2\,\mathbf{w}) = 0$.

13. A matrix is **Hermitian** if $S = S^{\mathrm{H}}$ and **anti-Hermitian** if $S = -S^{\mathrm{H}}$, where S^{H} is the conjugate transpose. Classify each matrix below as Hermitian, anti-Hermitian, or neither.

(a) $\begin{bmatrix}1 & 2+i\\2-i & 4\end{bmatrix}$ (b) $\begin{bmatrix}0 & 1-2i\\-1-2i & 3i\end{bmatrix}$ (c) $\begin{bmatrix}2 & 1+i\\1+i & 3\end{bmatrix}$

14. Let $S = \begin{bmatrix}2 & 1-i\\1+i & 3\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix}1\\i\end{bmatrix}$.

(a) Verify that $S = S^{\mathrm{H}}$.

(b) Compute $\mathbf{x}^{\mathrm{H}}\,S\,\mathbf{x}$ and check that the result is a real number.

The End.